Derivatives of Analytic Functions

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48. Derivatives of Analytic Functions

Remark :

It follows from the Cauchy Integral formula that if a function is analytic at a point, then its derivatives of all orders exist at that point and are themselves analytic there.

Lemma : Suppose that a function **f** is analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If **z** is any point interior to C, then

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^2} \quad \text{and} \quad f''(z) = \frac{1}{\pi i}\int_C \frac{f(s)ds}{(s-z)^3} \qquad \cdots\cdots \textbf{(1)}$$

Proof :

By Cauchy Integral formula

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(s)ds}{s - z} \qquad \cdots\cdots \textbf{(2)}$$

Where **z** is interior to C and **s** denotes points on C.

Let d = the smallest distance from Z to points on C.

Then

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{1}{2\pi i}\int_C \left(\frac{1}{s-z-\Delta z} - \frac{1}{s-z}\right)\frac{f(s)}{\Delta z}ds$$

$$= \frac{1}{2\pi i}\int_C \frac{f(s)ds}{(s-z-\Delta z)(s-z)}, \text{ where } 0<|\Delta z|<d$$

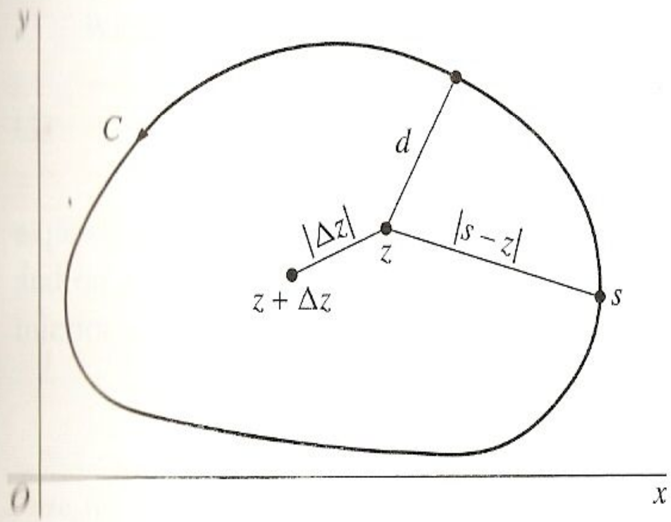

FIGURE 65

Then $\dfrac{f(z+\Delta z) - f(z)}{\Delta z} - \dfrac{1}{2\pi i}\displaystyle\int_C \dfrac{f(s)ds}{(s-z)^2}$

$$= \frac{1}{2\pi i}\int_C \frac{f(s)ds}{(s-z-\Delta z)(s-z)} - \frac{1}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^2}$$

$$= \frac{1}{2\pi i}\int_C \frac{(s-z-s+z+\Delta z)f(s)ds}{(s-z-\Delta z)(s-z)^2}$$

$$= \frac{1}{2\pi i}\int_C \frac{\Delta z\, f(s)ds}{(s-z-\Delta z)(s-z)^2} \quad \cdots\cdots \textbf{(3)}$$

Let M = The maximum value of $|f(s)|$ on C.

Note that $|s-z| \geq d$ and $|\Delta z| < d$.

Now $|s - z - \Delta z| = |(s - z) - \Delta z| \geq ||(s - z)| - |\Delta z|| \geq d - |\Delta z| > 0$

So $\left| \int_C \frac{\Delta z f(s) ds}{(s - z - \Delta z)(s - z)^2} \right| \leq \frac{|\Delta z| M}{(d - |\Delta z|) d^2} L$

Where L is the length of C.

$\to 0$ as $\Delta z \to 0$

$$\Rightarrow \frac{1}{2\pi i}\int_C \frac{\Delta z\, f(s)ds}{(s-z-\Delta z)(s-z)^2} \to 0 \text{ as } \Delta z \to 0$$

$$\Rightarrow \lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{1}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^2}$$ **[by (3)]**

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^2}$$

Differentiate w.r.t. **z**

$$f''(z) = \frac{1}{2\pi i}\frac{d}{dz}\left[\int_C \frac{f(s)ds}{(s - z)^2}\right] = \frac{1}{2\pi i}\int_C \frac{\partial}{\partial z}\frac{f(s)}{(s - z)^2}dz$$

$$= \frac{1}{2\pi i}\int_C \frac{(-2)(-1)\, f(s)}{(s - z)^3}\, dz = \frac{2\,!}{2\pi i}\int_C \frac{f(s)ds}{(s - z)^3}\, dz\, . \quad \blacksquare$$

Theorem 1 : If a function is analytic at a point, then its derivatives of all orders exist at that point. Those derivatives are, moreover, all analytic there.

Proof :

Given: **f** is analytic at z_0.

Then there exists a neighborhood $|z-z_0|<\in$ of z_0 throughout which **f** is analytic.

$\Rightarrow$ there is a positively oriented circle C_0, centred at z_0 and radius $\frac{\in}{2}$, such that **f** is analytic inside and on C_0

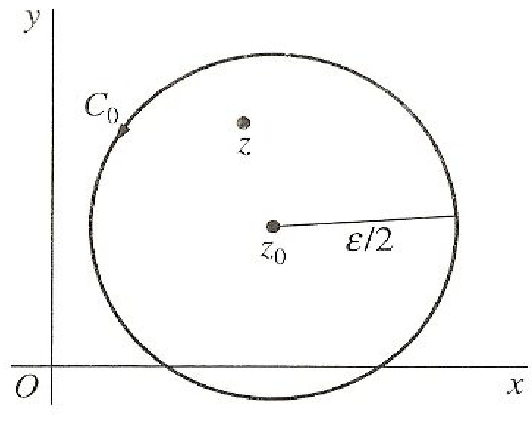

FIGURE 66

$\Rightarrow$ (by the above Lemma)

$$f''(z)=\frac{1}{\pi i}\int_{C_0}\frac{f(s)\,ds}{(s-z)^3}$$

at each point **z** interior to C_0.

Now the existence of $f''(z)$ throughout the neighborhood $|z-z_0|<\frac{\epsilon}{2}$ **means that** f^1 is analytic at z_0. We can apply the same argument to the analytic function f^1 to conclude that its derivative f^{11} is analytic. Continuing the same argument we can conclude that the derivatives of all orders exist and are analytic at z_0. ■

Corollary : If a function $f(z) = u(x,y) + iv(x,y)$ is defined and analytic at a point z = (x,y) then the component functions u and v have continuous partial derivatives of all orders at that point.

Proof :

Given : $f(z) = u(x,y) + iv(x,y)$ is analytic at a point z = (x,y).

$\Rightarrow$ f^1 is analytic at z.

$\Rightarrow$ $f'(z) = u_x + iv_x = v_y - iu_y$ is continuous at **z.**

$\Rightarrow$ The first order partial derivatives of u and v are continuous at **z**.

w.k: f' is analytic at z $\Rightarrow f''$ is analytic at **z.**

$\Rightarrow \; f''(z) = u_{xx} + iv_{xx} = v_{yx} - iu_{yx}$ is continuous at **z**.

$\Rightarrow$The second order partial derivatives of u and v are continuous at **z**.

Continuing the same argument we conclude that **u** and **v** have continuous partial derivatives of all orders at **z**. ■

Remark: Using mathematical induction we generalize the values of $f'(z)$ and $f''(z)$ to

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^{n+1}} \qquad (n = 1,2,\ldots\ldots.) \qquad \ldots\ldots \textbf{(4)}$$

Let $f^{(0)}(z) = f(z)$ and $0! = 1$

Then **(4)** is also valid for n = 0. We can write **(4)** as

$$\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(z_0) \qquad (n = 0,1,2,\ldots\ldots.)$$

This is useful in evaluating certain integrals when f is analytic inside and on a simple closed contour C, taken in the positive sense, and z_0 is any point interior to C. ■

Example : 1 If C is the positively oriented unit circle $|z|=1$ and f(z)=exp (2z) then find $\int_C \frac{\exp(2z)\,dz}{z^4}$

Solution: f(z) = exp (2z)

$$\int_C \frac{\exp(2z)\,dz}{z^4} = \int_C \frac{f(z)\,dz}{(z-0)^{3+1}} = \frac{2\pi i}{3!} f'''(0) = \frac{8\pi i}{3}$$

since $f'(z) = 2\exp(2z) \Rightarrow f^{11}(z) = 4\exp(2z)$

$\Rightarrow f'''(z) = 8\exp(2z)$

$\Rightarrow f'''(0) = 8$

Example : 2

Let z_0 be any point interior to a positively oriented simple closed contour C.

Find $\int_C \frac{dz}{(z - z_0)^{n+1}}$ (n = 0,1,2,…….).

Solution :

Let f(z) = 1 $\Rightarrow f^{(n)}(z) = 0$ (n = 1,2,…….).

So $f^{(n)}(z_0) = 0$ (n = 1,2,…….), But $f(z_0) = 1$. So $\int_C \frac{dz}{z - z_0} = 2\pi i$

We know $\int_C \frac{f(z)dz}{(z - z_0)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(z_0)$ (n = 0,1,2,…….).

So $\int_C \frac{dz}{(z - z_0)^{n+1}} = 0$ (n = 1,2,…….). ■